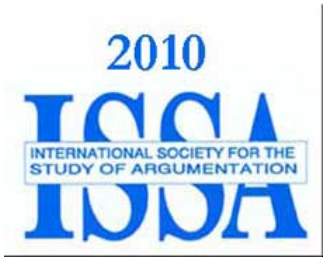
ISSA Proceedings 2010 – Didactical Arguments And Mathematical Proofs



There seems to be a mismatch between the classification of arguments given by Aristotle at the beginning of the *Sophistical Refutations* and some influential contemporary theories of argument for they do not pay much attention to a whole kind of Aristotelian arguments, namely didactical arguments.

An explanation could be that didactical arguments are implicitly included in these theories. But if you grant that didactical arguments differ from dialectical arguments in many respects and if you consider that for these theories the very notion of argument is dialectical, this interpretation of the demise of didactical arguments is not very plausible unless it results from equivocation on the word *"dialectical"*.
After a review of Aristotle's classification we shall examine these theories to see if they are well suited to accommodate the kind of argument Aristotle called didactical.

1. Aristotle's four arguments
In the *Sophistical Refutations* (II, 165a-b) Aristotle claims there are four kinds of διαλέγεσθαι λογων, an expression generally translated by "argument (or reasoning) involved in a discussion". This expression can also be interpreted simply as "dialogue" or "dialectic", taken in the broad sense of "talking together". Although Aristotle neither uses the word "syllogism" nor "enthymeme" it seems reasonable to agree with the translation using the word "argument" since the Philosopher stresses that these discourses have premises. And it is these premises which make the main difference between the four kinds of argument. In short:
Dialectical arguments are rooted in an endoxa, a common opinion.
Critical arguments start from premises accepted by the answerer but also granted by the arguer for his discourse aims at "showing that he [the arguer] knows".
Eristic arguments reason from premises that appear to be generally accepted but are not so.

Finally, didactical arguments do not reason from the opinions of the answerer but from "principles appropriate to each μαθηματος". Before commenting on this last word, it should be noted that, a few lines further, Aristotle says that dialectical, critical and eristic arguments are studied in specific books and "demonstrative" ones in the *Analytics*. Therefore, he holds didactical arguments to be demonstrative.

The word μαθημα is usually translated by "branch of knowledge" or "discipline" but it also means "lecture" or "lesson", two notions often related to an educational context. It is also close to μαθηματιcος which means "someone who studies" or "relative to a field of knowledge" and, of course, it is also germane to μαθηματιcα, usually translated by "mathematics".

Although it is demonstrative we should avoid to identify a didactical argument with what we now call a mathematical proof for the very notions of mathematics and science have changed since Aristotle. Remember, for instance, that he held sciences like optics, astronomy and music (harmony) to belong to the mathematical science even if pertaining to "more physical" parts of it (*Physics*, II, 2 194a). For Aristotle, what makes something "mathematical" is rather the way you consider it, namely the properties you drop in the process of abstraction and the principles you take into account, some of them being proper and some others not proper to the said science (*Posterior Analytics*, I, 10, 76a, 35-40). This is why one should take didactical argument to mean deductive argument based on the principles of a field of knowledge, of a discipline. It is "mathematical" in the broad sense of "systematic".

That the four types of arguments are "open to discussion" does not entail that they are always debatable. For Aristotle's definition of science requires that the conclusion of the arguments which are scientific to follow necessarily from their premises. And if these premises belong to the principles of a science they must be "true, primary, immediate, better known than, prior to, and causative of the conclusion" (*Posterior Analytics*, I, 2, 71b 20). Since their principles cannot be demonstrated but only grasped by induction – a specific act of abstraction – and their conclusion are necessary, Aristotelian scientific arguments are not "open to discussion" even if Aristotle grants that a superficial debate is always possible (*Posterior Analytics,* I, 10, 76b 25-30). A discussion may only occur in the case of postulates, namely *demonstrable* propositions supposed by the master but not by student.

2. What is left today?

I have just called to Aristotle's categorization to stress a contrast with some contemporary views about what counts as an argument. Today, dialectical, critical and eristic argumentations are well alive and acknowledged. The three of them are even key notions in distinct fields of investigation. But what happened to didactical arguments? They seem to have disappeared. How come that several of the prominent contemporary theories of argumentation do not consider them as specific arguments or even as genuine arguments?

This could be a consequence of a fundamental theoretical orientation. The revival of argumentation studies began around the mid-twentieth century with Perelman's and Toulmin's reactions against the infatuation of philosophy with formal logic. Perelman made an extra step by linking closely together the notions of science, rationality, demonstration, proof, certainty, logic and mathematics, a move which allowed him to build his empire of rhetoric against the world of proof, demonstration and certainty, including natural sciences and, first and foremost, mathematics. For instance, according to him, Descartes "considered as rational only demonstrations" (starting from clear ideas) and since the nineteenth century "under the influence of logicians-mathematicians, logic has been limited to formal logic, namely the study of the means of proof used in mathematical sciences" (Perelman 1958, p. 2-3).

Inspired or not by Perelman, many streams of contemporary argumentation studies have rooted their concept of argument into a broad notion of dialectic. And some scholars take for granted that proofs and arguments are different things: a proof is not a kind of argument or a part of an argument; it is no argument at all. Hence the view that argumentation is foreign to hard sciences and, especially, to mathematical demonstration.

This view, making an oxymoron of the notion of scientific argument has been challenged from various areas since a few decades (Lakatos, 1976; Finnochiaro 1980; Gross 1990) and the exclusion of mathematics from the kingdom of argumentation has been seriously challenged recently (Rav 1999; Dove 2007, 2009; Aberdein 2005, 2009). My own call to a reappraisal of Aristotelian didactical arguments wants to be another contribution to the refutation of the dogma of a sharp distinction between scientific demonstration and argumentation.

A pragmatic approach is certainly required by any theory of argumentation based

on the way people actually argue. But a systematic call to dialectic in the very definition of an argument results in an unfortunate narrowing of the field of study for it leaves out some argumentative forms, especially didactical arguments. The point is that it is possible to be an argument without being dialectical unless the very notion of dialectic is made so loose that it accommodates any argument. According to me, the fading of didactical arguments comes from a soft imperialism of dialectic.

What is meant by "dialectic"? As many old and tired words it has become vague and covers a range of different notions after an already equivocal career in the ancient times. In Aristotle, for instance, Wolf (2010, p 25-33) distinguishes three different meanings of "dialectic". Its broadest sense is "discussion" or "conversation": we have seen that the four kinds of arguments, including dialectical and didactical arguments, can be said dialectical in this sense. A second meaning is more specific since it refers to a regulated dialogue, typically between two participants. Paradigmatic examples are the dialectical debates at the core of Plato's dialogues or Aristotle's *Topica*. Finally, the narrowest definition is found in the *Sophistical Refutations*: "Dialectical arguments are those that reason from premises generally accepted, to the contradictory of a given thesis". Dialectic is here based on the endoxa and you can notice that this definition does not contain a term referring to an arguer or an opponent.

Nowadays, dialectical argumentation is usually not identified by the status of its premises but rather by its pragmatic goal, namely arguing *against* a thesis. Refutation, opposition or, at least, resistance are key notions in the contemporary understanding of dialectical argumentation which comes very close to controversy.
Many contemporary theories include a dialectical requirement in the very definition of an argument: if it does not go against the view of an explicit opponent, at least it supports a view against alternatives that could be held by opponents. I shall use the expression "virtual dialectic" to qualify a dialectical opposition which is only potential, that is which does not identify an actual opponent. From a logical point of view the conclusion of any argument opposes at least its negation and this makes any argument virtually dialectical. Hence, any theory accepting virtual dialectic as a genuine kind of dialectic can claim to be dialectical. This broadening of the notion of dialectic provides a concept wide enough to cover the whole field of argumentation: since not all arguments are

dialectical in a narrow sense, dialectic has to become virtual to accommodate any argument. But this broadening does not cancel the fact that didactical arguments belong to a field of knowledge where they are viewed as deductive and do not aim at a refutation. Their dialectical use is only derivative.

3. Dialectic accommodated

Pragma-dialectics claims that argumentation aims at the resolution of a difference of opinion by rational and critical means (Van Eemeren & Grootendorst, 2004). The basic disagreement may not be an open opposition: pragma-dialectics allows being an opponent without holding the contrary view. Sometimes, you argue with people who do not deny your position but only doubt. According to pragma-dialectics, such a situation can be qualified as dialectical. But it is not Aristotelian dialectic if the skeptic does not aim at a refutation of the proponent's thesis but only waits for convincing evidence. So, you can grant to pragma-dialectics that a difference of opinion does not always amount to a genuine divergence for some doubts are challenges and some are not. However, a different opinion *can* be looked upon as a kind of opposition, just like resistance or inertia can be interpreted as a form of opposition. But when your interlocutor's doubt does not challenge the rationality of your position, you do not argue against an active opponent but against someone who hesitates between several opinions. Ignorance too can be seen as a kind of opposition even if in some didactical contexts you do not argue with people who have a different opinion but with people who have no opinion at all. In such a case, as in the case of a non challenging doubt, the opposition is only potential. Pragma-dialectics will make a virtual dialectic out of a didactical situation whose specificity is not acknowledged since the interlocutor does not assume a critical position.

In *Manifest Rationality* Ralph Johnson holds that an argument has two sides, two tiers. One is the illative core, the fact that an argument is made of reasons supporting a thesis. And since this is not enough to account for the practice of argumentation, a dialectical tier is required. But this dialectical component does not imply an actual opposition between the arguers. Johnson writes: "that there is an argument, in the first place means that the conclusion is at least *potentially* controversial" (Johnson 2000, p. 206). Here again argumentation is made dialectic by means of a virtual dialectic. And it is the dialectical tier which makes a major difference between a mathematical proof and an argument for "No mathematical proof has or needs to have a dialectical tier" (Johnson 2000, p. 232). But is it

really sufficient to support the claim that a proof is not an argument? Can't a demonstration be "at least potentially controversial"? Some of them have been notoriously controversial, at least in their early days.

Johnson adds an interesting epistemic comment about the relationship between proof, argument and epistemic level. "The proof that there is no greatest prime number is conclusive, meaning that anyone *who knows anything about such matters***[i]** sees that the conclusion must be true for the reasons given" (Johnson 2000, p 232). In some way, this is certainly true. But on the one hand Johnson's view also suggests that in mathematics you would argue only with someone who does not stand on a sufficient epistemic footing and, on the other hand, that opposition is not possible between peers because all are convinced by the proof. This last idea of a necessary agreement between educated people reminds us Aristotle's thesis that scientific arguments are not open to discussion. But what happens with someone who only knows some things, not any thing, in the mathematical field and feels concerned by the question of a greatest prime number?

A dialectical treatment may not be possible here for, taken narrowly, dialectical argumentation presupposes a partial epistemic equality or symmetry between the arguers since it has to rely on common premises that may not be shared by anybody. (Remember Aristotle's formula about them: "they commend themselves to all or the majority or to the wise – that is, to all of the wise or to the majority or to the most famous and distinguished of them." (*Topica*, I, 1, 100, b 20)). So, what rational solution is left when you can't find common premises but you still want to argue that there is no greatest prime number? The authoritative use of didactical arguments which requires granting the truth of the proof premises. In some way, this is a means to make them common and, therefore, to reduce didactical argumentation to dialectic. But it also eliminates the specific cognitive context of didactical argumentation.

Not all reasoned dialogical forms at the core of Douglas Walton's conception of argumentation presuppose epistemic symmetry. According to him, informal logic brought a major contribution to the study of arguments by replacing them in the context of their utterance and he holds this context to be essentially conversational. He acknowledges a debt to Hamblin's notion of a dialectical system understood as "regulated dialogue" (Hamblin 1970, p. 232), that is several participants "speaking in turn in accordance with a set of rules or conventions"

(Hamblin 1970, p. 255). But are turns of speech essential to argumentation? The Aristotelian notion of a didactical argument has no such requirement: it may happen in a situation deprived of any turn of speech and so, it is only broadly dialectical. The character Aristotle calls the "answerer" may keep silent and even anonym during all the time of the transaction. This is not unusual: it is an ideal classroom situation, especially during a mathematical demonstration.

This quasi anonymity is even typical of didactical argumentation for, leaving aside eristic arguments, it is not possible with the other Aristotelian kinds of arguments for they have to be adjusted to the other party. In a critical argument the answerer cannot be anonymous since the premises of the argument are borrowed from him. This personal adjustment may seem less salient in the case of a dialectical argument since its premises do not come from the opponent but from common opinion. But when a dialectical argument is not only virtual, the arguer knows the person or the party she is talking to and chooses her common premises accordingly.

Contrary to the model at the core of pragma-dialectics which presupposes a critical symmetry between the arguers, Walton's approach leaves room for asymmetric epistemic situations. This is the case of information seeking dialogues. In *Informal Logic/ A Pragmatic Approach* Walton writes (Walton 1989, p. 7) that besides persuasion, inquiry and negotiation dialogues which are "the fundamental kinds or reasoned criticism", there are other forms including information-seeking dialogues. Here, "one party has the goal of finding information that the other party is believed to possess". This seems to come close to Aristotle's didactical arguments. However there is a difference stemming from Walton's dialogical/dialectical a priori. In an information-seeking dialogue the seeker is not the answerer but the questioner, the one who initiates the exchange. "The role of the respondent is to transmit the information by giving answers or replies that are as clear and helpful as possible" (Walton 1996, p. 126). On the contrary, a didactical argument does not require a previous question to be asked. This can be illustrated by the case of professors making demonstrations in front of students who do not ask any question. Such a context is pragmatic without being dialectical or dialogical, except in the broadest sense. Walton avoids the restrictive view limiting argument to controversy, but making any argument part of a dialectical/dialogical system keeps too restrictive for it fails to acknowledge the pragmatic peculiarities of didactical arguments.

We come more explicitly to the relation between virtual dialectic, didactical arguments and mathematical proof with Eric Krabbe (Krabbe, 2008). His view is inspired by the integrated version of pragma-dialectics and he grants that proofs can be involved in dialectical exchanges. But he does not assume that mathematical proofs are arguments. Like most people having paid attention to the practice of mathematicians, he resists the common temptation to reduce all their works and productions to proofs. A proof is only an object – often a goal – in the life of mathematics and mathematicians. Historians and mathematicians, among others Pólya (Pólya, 1945, 1954) and Lakatos (Lakatos, 1976), have already stressed that informal exchanges and dialectical argumentation is very common in mathematical research, notably during the stage that classical rhetoric dubbed the "invention" of a proof. Mathematicians are sometimes at pain finding the demonstration of a conjecture and they have to argue to go ahead. Sometimes one of them argues with himself. And when the time has come to present a proof to colleagues, argumentation may still be needed to convince them. History is full of corpses of failed or uncompleted demonstrations, convincing for a time or for no time.

Krabbe grants that mathematical proofs may have an argumentative dimension of their own, but he keeps within an a priori dialectical conception of argumentation. For instance, about the various kinds of discussions arising around proofs he writes: "they are argumentative in the sense that, given some difference or conflict, they serve to overcome the doubt of an interlocutor". And he adds: "whenever in a proof the reasoning displays persuasive functions, the proof *can***[ii]** be regarded as an argument" (Krabbe 2008, p. 457). Yes, it can. But persuasion is not always the result of a fight against an opposition or a doubt. If persuading amounts here to giving reasons to make someone believe something, a previous opposition or doubt may not be necessary. To have no opinion about a claim is both an opinion (a position) and a different opinion without being a doubt. You can persuade ignorant people too. And didactical arguments can do that.

Krabbe asks: "Is a formalized proof not the natural limit of dialectical depth"? Yes, but a limit touching two areas, different but close to each other and sometimes partly overlapping, the dialectical and the didactical one. Krabbe is certainly right when saying that "proof in a didactic context has not just explanatory functions, but also persuasive ones" (Krabbe 2008, p. 458). It may not be easy to disentangle one from the other, for understanding a proof is the result of both.

I neither contest what Krabbe says about dialectical situations in the practice of mathematics nor Johnson's claim that "the conclusion of an argument is at least potentially controversial". The assertion of the conclusion of an argument goes at least against contradictory statements but, *per se*, this trivial potential opposition does not require a pragmatic approach. Virtual dialectic can be seen as universal, but it lacks the pragmatic definiteness which makes an argumentation really contextual. And it has the drawback of concealing the specificity of didactical arguments or at least of a didactical use of arguments which requires neither an opposition nor an actual dialogue.

4. A thought experiment

Finally, here is an anecdote showing again that blurring the border between dialectic and didactic does not eliminate their specificities. It relies on two facts. First, that a mathematical demonstration has no definite length (We tend to forget it when talking about "*the*" demonstration of a theorem); second, that when you make a demonstration you sometimes "*jump*" from one statement to another, taking a shortcut that not everybody may follow.

During a public demonstration several voices broke the silence after a mathematician took a shortcut to reach his conclusion. One looked satisfied: "Yes. Brilliant! Very convincing." Another complained "Wait! How do you get to the conclusion from the previous step?" And a third voice went on: "Come on! You have not proven that unbelievable conclusion." Doubt, perhaps opposition, is creeping in with this last comment. But is the second one the expression of an opposition or a doubt? Not necessarily, it may be motivated by a lack of understanding.

Very devoted to his audience, the mathematician decided to give a single answer to everybody and began to get into the missing details. And at the same time his speech gave a proof, explained and argued. And even if the last voice, the dialectical one had not been heard, the improved support that the mathematician gave to his conclusion would still have been an argument.

NOTES

i My emphasis.

ii My emphasis.

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